Definitions

Definition 1 *Let us define the entropy of the finite valued random variable $X \sim p$ as*

$$H(p) = H(x) = -\sum_{i=1}^{L} p(x_i) log_2(p(x_i)),\ H(0) = 0.$$

Definition 2 *Suppose X and Y are random variables assuming values in finite sets. Then the quantity*

$$H(X|Y) = \sum_{i=1}^{L} H(X|Y = y_i)p(y_i)$$

is called the conditional entropy of the random variable X with respect to the random variable X.

Definition 3 *The typical set $A_\epsilon^n(X)$ is a set of sequences $x_1 x_2 \dots x_n$ such that*

$$e^{-nH(X)+\epsilon} \geq p_{X_1,X_2,\dots,X_n}(x_1 x_2 \dots x_n) \geq e^{-nH(X)-\epsilon}$$

Questions and problems 1.

1. Prove that:
   a) $H(p) \geq 0$ for all probability distribution p;
   b) $H(p) = 0$ if and only if p is degenerate distribution;
   c) $H(p) \leq log_2 L$ with equality if and only if p is the uniform distribution.
   d) $H(X|Y) \leq H(X)$;
   e) $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$.

2. Let X be a random variable having the probability distribution

   $$p(x) = \frac{1}{2^x},\ x = 1, 2, \dots$$

   Compute the entropy $H(X)$.

3. Select an integer $X \in U\{1, 2, \dots, n\}$ and an integer $Y|X = x \in U\{1, 2, \dots, n\}$. Compute the joint entropy $H(X,Y)$ without using the joint distribution of the random variable (X,Y).

4. Prove the following preposition:

   Preposition 1 *If $X_1, X_2, \dots, X_n$ are independent and identically distributed wit the distribution p, then*

   $$-\frac{1}{n} log_2 p_{X_1,X_2,\dots,X_n}(X_1, X_2, \dots, X_n) \to H(X)$$

   in probability.

5. Verify three properties of the typical set:

   a) if $(x_1 x_2 \dots x_n) \in A_\epsilon^n$, then

   $$P(X_1 = x_1, \dots, X_n = x_n) \propto e^{-nH(X) \mp \epsilon}$$

   b) $P(A_\epsilon^n) > 1 - \epsilon$ for n sufficiently large;

   (a) $|A_\epsilon^n| \geq e^{-nH(X)+\epsilon}$.