HMM for Bioinformatics

1. Let $\{X_n\}_{n=0}^{\infty}$ be a homogeneous Markov process:
   (a) prove that the Markov property implies that X_{n+1} and X_{n-1} are conditionally independent given X_n;
   (b) show that the conditional probabilities
   $$p_{ij}(n) = P(X_{m+n} = j | X_m = i),\ n \geq 1$$
   are independent of m;
   (c) show that
   $$P(X_{m+1} = j_{m+1}, \ldots, X_{m+n} = j_{m+n} | X_0 = j_0, \ldots, X_m = j_m) =$$
   $$P(X_{m+1} = j_{m+1}, \ldots, X_{m+n} = j_{m+n} | X_m = j_m);$$
2. Assume that there is a positive integer N such that
   $$p_{ij}(N) > 0$$
   for all i and j. Prove that if the vector π is a solution of
   $$\pi_j = \sum_{k=1}^{J} \pi_k p_{kj}, j = 1, \ldots, J$$
   then π is unique
3. Let $Z_1, \ldots, Z_n$ be a successive outcomes of rolling a die for any n. We consider $Z_1, \ldots, Z_n$. Furthermore
   $$S_n = Z_1 + \ldots + Z_n$$
   and
   $$X_n = S_n mod(5)$$
   which means that X_n is the remainder when S_n is divided by 5.
   (a) show that $\{X_n\}_{n=0}^{\infty}$ is a Markov process;
   (b) determine the probability transition matrix P for $\{X_n\}_{n=0}^{\infty}$;
   (c) determine the invariant distribution π for P;
4. The Markov process $\{X_n\}_{n=0}^{\infty}$ with the state space $S = \{1, 2, 3\}$ has the transition matrix
   $$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
   Compute P^{2n} and P^{2n-1} for all $n \geq 1$ and draw conclusions about the ergodicity of the chain.

5. Let

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

be a transition matrix

(a) verify that

$$P^n = \frac{1}{p+q}\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} + \frac{(1-p-q)^n}{p+q}\frac{1}{p+q}\begin{pmatrix} p & -p \\ -q & q \end{pmatrix}$$

(b) Find $\lim_{n\to\infty} P^n$.